Introduction to Mathematics and Modeling

Lecture 4

Laws of differentiation
1. Section 3.3: laws of differentiation
2. Section 3.3: derivatives of exponential functions
3. Section 3.5: derivatives of trigonometric functions
Basic functions:

- **Power functions:** $x^\alpha$
- **Trigonometric functions:** $\sin(x), \cos(x), \tan(x)$
- **Exponential functions:** $a^x, e^x$
- **Logarithms:** $\log_a x, \ln x$ (next lecture)

Rules:

- **Constant multiples**
- **Sums**
- **Products**
- **Quotients**
- **Compositions** ('Chain Rule': next lecture)
The derivative of \( f(x) = x^\alpha \)

Let \( \alpha \) be a real number and let \( f(x) = x^\alpha \) then

\[
f'(x) = \alpha x^{\alpha - 1}
\]
Differentiating powers

**The derivative of** \( f(x) = x^\alpha \)

*Let \( \alpha \) be a real number and let \( f(x) = x^\alpha \) then*

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The derivative of \( f(x) = x^\alpha \)

Let \( \alpha \) be a real number and let \( f(x) = x^\alpha \) then

\[
f'(x) = \alpha x^{\alpha - 1}
\]

Examples:

\[
f(x) = 1 = x^0 \quad \implies \quad f'(x) = 0 \cdot x^{-1} = 0
\]
The derivative of \( f(x) = x^\alpha \)

Let \( \alpha \) be a real number and let \( f(x) = x^\alpha \) then

\[
f'(x) = \alpha x^{\alpha-1}
\]

Examples:

\[
f(x) = 1 = x^0 \implies f'(x) = 0 \cdot x^{-1} = 0
\]

\[
f(x) = x = x^1 \implies f'(x) = 1 \cdot x^0 = 1
\]
The derivative of $f(x) = x^\alpha$

Let $\alpha$ be a real number and let $f(x) = x^\alpha$ then

$$f'(x) = \alpha x^{\alpha-1}$$

Examples:

$$f(x) = 1 = x^0 \quad \implies \quad f'(x) = 0 \cdot x^{-1} = 0$$

$$f(x) = x = x^1 \quad \implies \quad f'(x) = 1 \cdot x^0 = 1$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \quad \implies \quad f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$
The derivative of \( f(x) = x^\alpha \)

Let \( \alpha \) be a real number and let \( f(x) = x^\alpha \) then

\[
f'(x) = \alpha x^{\alpha-1}
\]

Examples:

\[
f(x) = 1 = x^0 \quad \implies \quad f'(x) = 0 \cdot x^{-1} = 0
\]

\[
f(x) = x = x^1 \quad \implies \quad f'(x) = 1 \cdot x^0 = 1
\]

\[
f(x) = \sqrt{x} = x^{\frac{1}{2}} \quad \implies \quad f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}
\]

\[
f(x) = \frac{1}{x} = x^{-1} \quad \implies \quad f'(x) = (-1) x^{-2} = -\frac{1}{x^2}
\]
Let $c$ be a constant, then

$$\frac{d}{dx}(c f(x)) = c f'(x).$$
Constant multiplication

Let $c$ be a constant, then

$$\frac{d}{dx} (c f(x)) = c f'(x).$$

Examples

$$\frac{d}{dx} (2x^4) = 2 \frac{d}{dx} (x^4) = 2 \cdot 4x^3 = 8x^3.$$
Let $c$ be a constant, then

$$
\frac{d}{dx}(c f(x)) = c f'(x).
$$

**Examples**

- \[
\frac{d}{dx}(2x^4) = 2 \frac{d}{dx}(x^4) = 2 \cdot 4x^3 = 8x^3.
\]

- \[
\frac{d}{dx}(2x)^4 = \frac{d}{dx}(2^4 x^4) = 16 \frac{d}{dx}(x^4) = 16 \cdot 4x^3 = 64x^3.
\]
Constant multiplication

Let \( c \) be a constant, then

\[
\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x).
\]

Examples

\[
\frac{d}{dx}(2x^4) = 2 \cdot \frac{d}{dx}(x^4) = 2 \cdot 4x^3 = 8x^3.
\]

\[
\frac{d}{dx}(2x)^4 = \frac{d}{dx}(2^4 \cdot x^4) = 16 \cdot \frac{d}{dx}(x^4) = 16 \cdot 4x^3 = 64x^3.
\]

\[
\frac{d}{dx}\sqrt{3x} = \frac{d}{dx}(\sqrt{3} \cdot \sqrt{x}) = \sqrt{3} \cdot \frac{d}{dx}\sqrt{x} = \frac{\sqrt{3}}{2\sqrt{x}}.
\]
The sum rule

For all functions $f$ and $g$ we have

$$
\frac{d}{dx} (f(x) + g(x)) = f'(x) + g'(x).
$$
The sum rule

For all functions $f$ and $g$ we have

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$

This rule can be extended to sums of more than 2 functions:

$$\frac{d}{dx}(f_1(x) + \cdots + f_n(x)) = f_1'(x) + \cdots + f_n'(x).$$
The sum rule

For all functions $f$ and $g$ we have

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$

- This rule can be extended to sums of more than 2 functions:
  $$\frac{d}{dx}(f_1(x) + \cdots + f_n(x)) = f_1'(x) + \cdots + f_n'(x).$$
- Notice that $f(x) - g(x) = f(x) + (-1)g(x)$. By applying the constant multiple rule and the sum rule we have the difference rule:
  $$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x).$$
\[
\frac{d}{dx}(x^3 + \frac{4}{3}x^2 - 5x + 1) = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)
\]
\[
\frac{d}{dx}(x^3 + \frac{4}{3}x^2 - 5x + 1) = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)
\]

\[
= 3x^2 + \frac{4}{3}\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x) + 0
\]
Example

\[
\frac{d}{dx}(x^3 + \frac{4}{3}x^2 - 5x + 1) = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)
\]

\[
= 3x^2 + \frac{4}{3} \frac{d}{dx}(x^2) - 5 \frac{d}{dx}(x) + 0
\]

\[
= 3x^2 + \frac{4}{3} \cdot 2x - 5 \cdot 1
\]
Example

\[
\frac{d}{dx}(x^3 + \frac{4}{3}x^2 - 5x + 1) = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)
\]

\[
= 3x^2 + \frac{4}{3}\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x) + 0
\]

\[
= 3x^2 + \frac{4}{3} \cdot 2x - 5 \cdot 1
\]

\[
= 3x^2 + \frac{8}{3}x - 5.
\]
\[
\frac{d}{dx} \left( \sqrt{x} + \frac{1}{4x^2} - x^{\frac{1}{3}} \right) = \frac{d}{dx} \left( x^{\frac{1}{2}} + \frac{1}{4} x^{-2} - x^{\frac{1}{3}} \right)
\]
Example

\[
\frac{d}{dx} \left( \sqrt{x} + \frac{1}{4x^2} - x^{\frac{1}{3}} \right) = \frac{d}{dx} \left( x^{\frac{1}{2}} + \frac{1}{4} x^{-2} - x^{\frac{1}{3}} \right)
\]

\[
= \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{4} (-2) x^{-3} - \frac{1}{3} x^{-\frac{2}{3}}
\]
Example

\[
\frac{d}{dx} \left( \sqrt{x} + \frac{1}{4x^2} - x^{\frac{1}{3}} \right) = \frac{d}{dx} \left( x^{\frac{1}{2}} + \frac{1}{4} x^{-2} - x^{\frac{1}{3}} \right)
\]

\[
= \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{4} (-2) x^{-3} - \frac{1}{3} x^{-\frac{2}{3}}
\]

\[
= \frac{1}{2\sqrt{x}} - \frac{1}{2x^3} - \frac{1}{3\sqrt[3]{x^2}}.
\]
Assignment: IMM1 - Tutorial 4.1
The product rule

**Theorem**

If $f$ and $g$ are differentiable at $x$ then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$
Theorem

If \( f \) and \( g \) are differentiable at \( x \) then

\[
\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).
\]

- If \( h \) is small then

\[
f(x + h) - f(x) \approx h f'(x) \quad \text{and} \quad g(x + h) - g(x) \approx h g'(x).
\]
The product rule

**Theorem**

If \( f \) and \( g \) are differentiable at \( x \) then

\[
\frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x).
\]

- If \( h \) is small then
  \[
f(x + h) - f(x) \approx h f'(x) \quad \text{and} \quad g(x + h) - g(x) \approx h g'(x).
  \]
- \[
\frac{(x + h)g(x + h) - f(x)g(x)}{h} = \frac{\text{A} + \text{B} + \text{C}}{h}
\]

Taking the limit \( h \to 0 \) gives the desired result.
Theorem

If \( f \) and \( g \) are differentiable at \( x \) then

\[
\frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x).
\]

- If \( h \) is small then

\[
f(x + h) - f(x) \approx h f'(x) \quad \text{and} \quad g(x + h) - g(x) \approx h g'(x).
\]

- \[
\frac{f(x + h)g(x + h) - f(x)g(x)}{h} = \frac{A + B + C}{h}
\]

\[
\approx \frac{hf'(x) \cdot g(x) + f(x) \cdot hg'(x) + h^2f'(x)g'(x)}{h}
\]
**Theorem**

If $f$ and $g$ are differentiable at $x$ then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

- If $h$ is small then

  $$f(x + h) - f(x) \approx hf'(x) \quad \text{and} \quad g(x + h) - g(x) \approx hg'(x).$$

- \[
  \frac{f(x + h)g(x + h) - f(x)g(x)}{h} \approx \frac{Kf'(x) \cdot g(x) + f(x) \cdot Kg'(x) + h^2f'(x)g'(x)}{h}
  \]

  \[
  = f'(x)g(x) + f(x)g'(x) + hf'(x)g'(x).
  \]
The product rule

**Theorem**

If $f$ and $g$ are differentiable at $x$ then

$$
\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).
$$

- If $h$ is small then

  $$
f(x + h) - f(x) \approx h f'(x) \quad \text{and} \quad g(x + h) - g(x) \approx h g'(x).
  $$

- \[ \frac{f(x + h)g(x + h) - f(x)g(x)}{h} = \frac{A + B + C}{h} \]

  \[ \approx \frac{hf'(x) \cdot g(x) + f(x) \cdot hg'(x) + h^2f'(x)g'(x)}{h} \]

  \[ = f'(x)g(x) + f(x)g'(x) + hf'(x)g'(x). \]

- Taking the limit $h \to 0$ gives the desired result.
Example

Differentiate \((x + 1)(x - 1)\).
### Example

**Differentiate** \((x + 1)(x - 1)\).

- Apply the product rule:

\[
\frac{d}{dx} (x + 1)(x - 1) = \left( \frac{d}{dx} (x + 1) \right) (x - 1) + (x + 1) \left( \frac{d}{dx} (x - 1) \right)
\]

\[
= 1 \cdot (x - 1) + (x + 1) \cdot 1
\]

\[
= x - 1 + x + 1 = 2x.
\]
Example

Differentiate $(x + 1)(x - 1)$.

- Apply the product rule:
  \[
  \frac{d}{dx} (x + 1)(x - 1) = \left( \frac{d}{dx} (x + 1) \right) (x - 1) + (x + 1) \left( \frac{d}{dx} (x - 1) \right)
  = 1 \cdot (x - 1) + (x + 1) \cdot 1
  = x - 1 + x + 1 = 2x.
  \]

- Alternatively we can expand $(x + 1)(x - 1)$:
  \[
  \frac{d}{dx} (x + 1)(x - 1) = \frac{d}{dx} (x^2 - 1)
  = 2x - 0 = 2x.
  \]
The reciprocal rule

**Theorem**

If $g$ is differentiable at $x$ then

$$
\frac{d}{dx} \left( \frac{1}{g(x)} \right) = -\frac{g'(x)}{g(x)^2}.
$$
The reciprocal rule

**Theorem**

*If* $g$ is differentiable at $x$ then

$$\frac{d}{dx} \left( \frac{1}{g(x)} \right) = -\frac{g'(x)}{g(x)^2}.$$

- Define $f(x) = 1/g(x)$, then

$$f(x)g(x) = 1$$
The reciprocal rule

Theorem

If $g$ is differentiable at $x$ then

$$
\frac{d}{dx} \left( \frac{1}{g(x)} \right) = -\frac{g'(x)}{g(x)^2}.
$$

- Define $f(x) = 1/g(x)$, then

$$
f(x)g(x) = 1
$$

$$
f'(x)g(x) + f(x)g'(x) = 0
$$

product rule
The reciprocal rule

**Theorem**

If $g$ is differentiable at $x$ then

$$
\frac{d}{dx} \left( \frac{1}{g(x)} \right) = - \frac{g'(x)}{g(x)^2}.
$$

- Define $f(x) = 1/g(x)$, then

  $$
  f(x)g(x) = 1
  \quad \text{product rule}
  $$

  $$
  f'(x)g(x) + f(x)g'(x) = 0
  $$

  $$
  f'(x)g(x) = -f(x)g'(x) = -\frac{g'(x)}{g(x)}
  $$
The reciprocal rule

Theorem

If $g$ is differentiable at $x$ then

$$
\frac{d}{dx} \left( \frac{1}{g(x)} \right) = - \frac{g'(x)}{g(x)^2}.
$$

- Define $f(x) = 1/g(x)$, then

\[
\begin{align*}
    f(x)g(x) &= 1 \\
    f'(x)g(x) + f(x)g'(x) &= 0 \\
    f'(x)g(x) &= -f(x)g'(x) = -\frac{g'(x)}{g(x)} \\
    f'(x) &= -\frac{g'(x)}{g(x)^2}.
\end{align*}
\]
The reciprocal rule

**Theorem**

*If* \( g \) *is differentiable at* \( x \) *then*

\[
\frac{d}{dx} \left( \frac{1}{g(x)} \right) = -\frac{g'(x)}{g(x)^2}.
\]

- Define \( f(x) = 1/g(x) \), then

\[
f(x)g(x) = 1
\]

\[
f'(x)g(x) + f(x)g'(x) = 0
\]

\[
f'(x)g(x) = -f(x)g'(x) = -\frac{g'(x)}{g(x)}
\]

\[
f'(x) = -\frac{g'(x)}{g(x)^2}.
\]

- Example: \[
\frac{d}{dx} \left( \frac{1}{x^2 + 1} \right) = -\frac{d}{dx} \frac{x^2 + 1}{(x^2 + 1)^2} = -\frac{2x}{(x^2 + 1)^2}.
\]
The quotient rule

**Theorem**

If $f$ and $g$ are differentiable at $x$, and $g(x) \neq 0$, then

$$
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
$$
The quotient rule

**Theorem**

If $f$ and $g$ are differentiable at $x$, and $g(x) \neq 0$, then

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\]

- The quotient rule can be proved with the reciprocal rule and the product rule (see exercises).
The quotient rule

**Theorem**

If $f$ and $g$ are differentiable at $x$, and $g(x) \neq 0$, then

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\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
$$

- The quotient rule can be proved with the reciprocal rule and the product rule (see exercises).
- Example:

$$
\frac{d}{dx} \left( \frac{x - 1}{x + 1} \right) = \frac{\left( \frac{d}{dx}(x - 1) \right)(x + 1) - (x - 1) \left( \frac{d}{dx}(x + 1) \right)}{(x + 1)^2}
$$
### Theorem

If \( f \) and \( g \) are differentiable at \( x \), and \( g(x) \neq 0 \), then

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
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- Example:

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\frac{d}{dx} \left( \frac{x-1}{x+1} \right) = \frac{\left( \frac{d}{dx}(x-1) \right)(x+1) - (x-1) \left( \frac{d}{dx}(x+1) \right)}{(x+1)^2}
\]

\[
= \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2}
\]
The quotient rule

**Theorem**

If \( f \) and \( g \) are differentiable at \( x \), and \( g(x) \neq 0 \), then

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\]

- The quotient rule can be proved with the reciprocal rule and the product rule (see exercises).
- Example:

\[
\frac{d}{dx} \left( \frac{x - 1}{x + 1} \right) = \frac{\left( \frac{d}{dx}(x - 1) \right)(x + 1) - (x - 1)\left( \frac{d}{dx}(x + 1) \right)}{(x + 1)^2}
\]

\[
= \frac{1 \cdot (x + 1) - (x - 1) \cdot 1}{(x + 1)^2}
\]

\[
= \frac{x + 1 - x + 1}{(x + 1)^2}
\]
The quotient rule

**Theorem**

If \( f \) and \( g \) are differentiable at \( x \), and \( g(x) \neq 0 \), then

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\]

- The quotient rule can be proved with the reciprocal rule and the product rule (see exercises).
- Example:

\[
\begin{align*}
\frac{d}{dx} \left( \frac{x - 1}{x + 1} \right) &= \frac{\left( \frac{d}{dx}(x - 1) \right)(x + 1) - (x - 1) \left( \frac{d}{dx}(x + 1) \right)}{(x + 1)^2} \\
&= \frac{1 \cdot (x + 1) - (x - 1) \cdot 1}{(x + 1)^2} \\
&= \frac{x + 1 - x + 1}{(x + 1)^2} \\
&= \frac{2}{(x + 1)^2}.
\end{align*}
\]
The quotient rule

**Theorem**

If $f$ and $g$ are differentiable at $x$, and $g(x) \neq 0$, then

$$
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
$$

- The quotient rule can be proved with the reciprocal rule and the product rule (see exercises).
- Example:

  $$
  \frac{d}{dx} \left( \frac{x - 1}{x + 1} \right) = \frac{\left( \frac{d}{dx}(x - 1) \right)(x + 1) - (x - 1) \left( \frac{d}{dx}(x + 1) \right)}{(x + 1)^2}
  \quad = \frac{1 \cdot (x + 1) - (x - 1) \cdot 1}{(x + 1)^2}
  \quad = \frac{x + 1 - x + 1}{(x + 1)^2}
  \quad = \frac{2}{(x + 1)^2}.
  $$
Exercises

- Prove the quotient rule with the product rule and the reciprocal rule by writing

\[
\frac{f(x)}{g(x)} = f(x) \cdot \left( \frac{1}{g(x)} \right)
\]

Assignment: IMM1 - Tutorial 4.2
The derivative of exponential functions

Define $k_a$ as the slope of the tangent line to the graph of $f(x) = a^x$ in the point $(0, 1)$, then

$$k_a = f'(0).$$
The derivative of exponential functions

Define $k_a$ as the slope of the tangent line to the graph of $f(x) = a^x$ in the point $(0, 1)$, then

$$k_a = f'(0).$$
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Define $k_a$ as the slope of the tangent line to the graph of $f(x) = a^x$ in the point $(0, 1)$, then

$$k_a = f'(0).$$
Let $f(x) = a^x$, then

$$k_a = f'(0)$$

$$= \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}.$$
Let $f(x) = a^x$, then

$$k_a = f'(0) = \lim_{h \to 0} \frac{a^0 + h - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}.$$ 

For arbitrary $x$ we have

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}.$$
The derivative of exponential functions

- Let \( f(x) = a^x \), then

\[ k_a = f'(0) \]

\[ = \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}. \]

- For arbitrary \( x \) we have

\[ f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \]

\[ = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \]
The derivative of exponential functions

Let \( f(x) = a^x \), then
\[
k_a = f'(0)
\]
\[
= \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}.
\]

For arbitrary \( x \) we have
\[
f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}
\]
\[
= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}
\]
\[
= \lim_{h \to 0} a^x \frac{a^h - 1}{h} = a^x k_a = k_a f(x).
\]
Let \( f(x) = a^x \), then

\[
k_a = f'(0) = \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}.\]

For arbitrary \( x \) we have

\[
f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} a^x \frac{a^h - 1}{h} = a^x k_a = k_a f(x).
\]

The derivative of an exponential function is \textit{proportional} to the function itself.
There is exactly one value of $a$ for which $k_a = 1$. 
There is exactly one value of $a$ for which $k_a = 1$.

This value is denoted as $e$ and it is approximately equal to 2.72.
There is exactly one value of \( a \) for which \( k_a = 1 \).

This value is denoted as \( e \) and it is approximately equal to 2.72.

More precise

\[
e \approx 2.7182818284590452353602874713527...
\]
The function $e^x$ is called the (natural) exponential function.
The derivative of exponential functions

- The function \( e^x \) is called the **(natural) exponential function**.
- It has the elegant property

\[
\frac{d}{dx} (e^x) = e^x
\]

The exponential function is its own derivative!
The function $e^x$ is called the **(natural) exponential function**.

- It has the elegant property
  \[
  \frac{d}{dx}(e^x) = e^x
  \]
  The exponential function is its own derivative!
- The number $e$ is used as the base for the **natural logarithm**:
  \[
  \ln(x) = \log_e(x).
  \]
The derivative of exponential functions

- The function $e^x$ is called the (natural) exponential function.
- It has the elegant property

\[
\frac{d}{dx} (e^x) = e^x
\]

The exponential function is its own derivative!
- The number $e$ is used as the base for the natural logarithm:

\[\ln(x) = \log_e(x).\]
- For the number $k_a$ the following holds:

\[k_a = \ln a.\]

To prove this you need the chain rule (next lecture).
The function $e^x$ is called the (natural) exponential function.

- It has the elegant property

  $$\frac{d}{dx} (e^x) = e^x$$

  The exponential function is it's own derivative!

- The number $e$ is used as the base for the natural logarithm:

  $$\ln(x) = \log_e(x).$$

- For the number $k_a$ the following holds:

  $$k_a = \ln a.$$  

  To prove this you need the chain rule (next lecture).

- We can now differentiate all exponential functions:

  $$\frac{d}{dx} (a^x) = \ln a \cdot a^x$$
Prove that \( \frac{d}{dx} (e^{ax}) = a \ e^{ax} \).

Hint: use the identity \( e^{ax} = (e^a)^x \).

Assignment: IMM1 - Tutorial 4.3
The sample function (denoted as \( \text{sinc} \)) is defined by

\[
\text{sinc}(x) = \begin{cases} 
\frac{\sin x}{x} & \text{if } x \neq 0, \\
1 & \text{if } x = 0.
\end{cases}
\]

The sample function is continuous at 0:

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]

Proof: Section 2.4; theorem 7, page 89
Area $S = \frac{1}{2} \theta r^2$.

Area $T = \frac{1}{2} ab$. 
The sample function

\[ A = \frac{1}{2} \cos \theta \sin \theta \]

\[ B = \frac{1}{2} \theta \]

\[ C = \frac{1}{2} \tan \theta = \frac{1}{2} \frac{\sin \theta}{\cos \theta} \]
The sample function

\[ A = \frac{1}{2} \cos \theta \sin \theta \]

\[ A < B \]

\[ B = \frac{1}{2} \theta \]

\[ B < C \]

\[ C = \frac{1}{2} \tan \theta = \frac{1}{2} \frac{\sin \theta}{\cos \theta} \]
The sample function

\[ A = \frac{1}{2} \cos \theta \sin \theta \]

\[ A < B \]

\[ B = \frac{1}{2} \theta \]

\[ B < C \]

\[ C = \frac{1}{2} \tan \theta = \frac{1}{2} \frac{\sin \theta}{\cos \theta} \]

\[ \sin \theta < \frac{1}{\cos \theta} \]

\[ \cos \theta < \frac{\sin \theta}{\theta} < \frac{1}{\cos \theta} \]

\[ \frac{\sin \theta}{\theta} > \cos \theta \]
The sample function

\[ A = \frac{1}{2} \cos \theta \sin \theta \]

\[ A < B \quad \frac{\sin \theta}{\theta} < \frac{1}{\cos \theta} \]

\[ B = \frac{1}{2} \theta \]

\[ B < C \quad \frac{\sin \theta}{\theta} > \cos \theta \]

\[ C = \frac{1}{2} \tan \theta = \frac{1}{2} \frac{\sin \theta}{\cos \theta} \]

If \( \theta \to 0 \), then \( \frac{\sin \theta}{\theta} \to 1 \)
Another trigonometric limit

Define \( g \) by

\[
g(x) = \begin{cases} 
  \frac{\cos x - 1}{x} & \text{if } x \neq 0, \\
  0 & \text{if } x = 0.
\end{cases}
\]

The function \( g \) is continuous at 0:

\[
\lim_{{x \to 0}} \frac{\cos x - 1}{x} = 0.
\]

Proof:

Section 2.4 example 5(a)
The derivative of \( \sin x \)

Define \( f(x) = \sin(x) \) then

\[
f(x + h) = \sin(x + h) = \sin(x) \cos(h) + \cos(x) \sin(h).
\]
The derivative of $\sin x$

Define $f(x) = \sin(x)$ then

$$f(x + h) = \sin(x + h)$$

$$= \sin(x) \cos(h) + \cos(x) \sin(h).$$

$$\frac{f(x + h) - f(x)}{h} = \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h}$$
Define $f(x) = \sin(x)$ then

$$f(x + h) = \sin(x + h)$$

$$= \sin(x) \cos(h) + \cos(x) \sin(h).$$

$$\frac{f(x + h) - f(x)}{h} = \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h}$$

$$= \sin(x) \frac{\cos(h) - 1}{h} + \cos(x) \frac{\sin(h)}{h}.$$
Define $f(x) = \sin(x)$ then
\[
f(x + h) = \sin(x + h) = \sin(x) \cos(h) + \cos(x) \sin(h).
\]

\[
\frac{f(x + h) - f(x)}{h} = \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h}
\]
\[
= \sin(x) \frac{\cos(h) - 1}{h} + \cos(x) \frac{\sin(h)}{h}.
\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}
\]
\[
= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \boxed{\cos(x)}.
\]
The derivative of $\cos x$

Define $f(x) = \cos(x)$ then

\[ f(x + h) = \cos(x + h) \]

\[ = \cos(x) \cos(h) - \sin(x) \sin(h). \]
Define $f(x) = \cos(x)$ then

$$f(x + h) = \cos(x + h)$$

$$= \cos(x) \cos(h) - \sin(x) \sin(h).$$

$$\frac{f(x + h) - f(x)}{h} = \frac{\cos(x) \cos(h) - \sin(x) \sin(h) - \cos(x)}{h}$$
Define \( f(x) = \cos(x) \) then

\[
f(x + h) = \cos(x + h) = \cos(x) \cos(h) - \sin(x) \sin(h).
\]

\[
\frac{f(x + h) - f(x)}{h} = \frac{\cos(x) \cos(h) - \sin(x) \sin(h) - \cos(x)}{h}
\]

\[
= \cos(x) \frac{\cos(h) - 1}{h} - \sin(x) \frac{\sin(h)}{h}.
\]
Define \( f(x) = \cos(x) \) then

\[
f(x + h) = \cos(x + h)
= \cos(x) \cos(h) - \sin(x) \sin(h).
\]

\[
\frac{f(x + h) - f(x)}{h} = \frac{\cos(x) \cos(h) - \sin(x) \sin(h) - \cos(x)}{h}
= \cos(x) \frac{\cos(h) - 1}{h} - \sin(x) \frac{\sin(h)}{h}.
\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
= \cos(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h \to 0} \frac{\sin(h)}{h}
= \cos(x) \cdot 0 - \sin(x) \cdot 1 = -\sin(x).
\]
The derivative of \( \sin x \) and \( \cos x \)

\[
\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x
\]
The derivative of $\tan x$

- Use the identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and the quotient rule:

\[
\frac{d}{dx} \tan(x) = \frac{d}{dx} \left(\frac{\sin(x)}{\cos(x)}\right)
\]
The derivative of $\tan x$

- Use the identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and the quotient rule:

$$
\frac{d}{dx} \tan(x) = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right)
$$

$$
= \frac{d}{dx} (\sin x) \cdot \cos x - \sin x \frac{d}{dx} (\cos x)
$$

$$
= \frac{\cos^2 x}{\cos^2 x}
$$
The derivative of $\tan x$

- Use the identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and the quotient rule:

$$\frac{d}{dx} \tan(x) = \frac{d}{dx} \left( \frac{\sin(x)}{\cos(x)} \right)$$

$$= \frac{d}{dx} (\sin x) \cdot \cos x - \sin x \frac{d}{dx} \cos x$$

$$= \cos x \cdot \cos x - \sin x (-\sin x)$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$
The derivative of $\tan x$

- Use the identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and the quotient rule:

\[
\frac{d}{dx} \tan(x) = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right)
\]

\[
= \frac{\frac{d}{dx}(\sin x) \cdot \cos x - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}
\]

\[
= \frac{\cos x \cdot \cos x - \sin(-\sin x)}{\cos^2 x}
\]

\[
= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}
\]

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The derivative of $\tan x$

\[
\frac{d}{dx} (\tan x) = \frac{1}{\cos^2 x}
\]
The derivatives of trigonometric functions

\[
\frac{d}{dx}(\sin x) = \cos x
\]

\[
\frac{d}{dx}(\cos x) = -\sin x
\]

\[
\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}
\]

Learn them by heart!...
Example

\[ f(x) = e^x \sin(x) \]

\[ f'(x) = \frac{d}{dx} (e^x) \cdot \sin(x) + e^x \cdot \frac{d}{dx} \sin(x) \]
Example

\[ f(x) = e^x \sin(x) \]

\[ f'(x) = \frac{d}{dx} (e^x) \cdot \sin(x) + e^x \cdot \frac{d}{dx} \sin(x) \]

\[ = e^x \sin(x) + e^x \cos(x) \]
Example

\[ f(x) = e^x \sin(x) \]

\[ f'(x) = \frac{d}{dx}(e^x) \cdot \sin(x) + e^x \cdot \frac{d}{dx} \sin(x) \]

\[ = e^x \sin(x) + e^x \cos(x) \]

\[ = e^x(\sin(x) + \cos(x)) \]
Example

\[ f(x) = \sin(2x) \]

\[ = 2 \sin(x) \cos(x) \]
Example

\[ f(x) = \sin(2x) \]

\[ = 2 \sin(x) \cos(x) \]

\[ f'(x) = 2 \left[ \frac{d}{dx} (\sin x \cos x) + \sin x \frac{d}{dx} (\cos x) \right] \]
Example

\[ f(x) = \sin(2x) \]

\[ = 2 \sin(x) \cos(x) \]

\[ f'(x) = 2 \left[ \frac{d}{dx}(\sin x) \cos x + \sin x \frac{d}{dx}(\cos x) \right] \]

\[ = 2 \left[ \cos x \cdot \cos x + \sin x \cdot (-\sin x) \right] \]
Example

\[ f(x) = \sin(2x) \]

\[ = 2 \sin(x) \cos(x) \]

\[ f'(x) = 2 \left[ \frac{d}{dx}(\sin x) \cos x + \sin x \frac{d}{dx}(\cos x) \right] \]

\[ = 2 \left[ \cos x \cdot \cos x + \sin x \cdot (- \sin x) \right] \]

\[ = 2 \left[ \cos^2 x - \sin^2 x \right] \]
Example

\[ f(x) = \sin(2x) \]
\[ = 2 \sin(x) \cos(x) \]

\[ f'(x) = 2 \left[ \frac{d}{dx}(\sin x) \cos x + \sin x \frac{d}{dx}(\cos x) \right] \]
\[ = 2 \left[ \cos x \cdot \cos x + \sin x \cdot ( - \sin x) \right] \]
\[ = 2 \left[ \cos^2 x - \sin^2 x \right] \]
\[ = 2 \cos(2x) \]
Example

\[ f(x) = \sin(2x) \]
\[ = 2 \sin(x) \cos(x) \]

\[ f'(x) = 2 \left[ \frac{d}{dx}(\sin x) \cos x + \sin x \frac{d}{dx}(\cos x) \right] \]
\[ = 2 \left[ \cos x \cdot \cos x + \sin x \cdot ( - \sin x) \right] \]
\[ = 2 \left[ \cos^2 x - \sin^2 x \right] \]
\[ = 2 \cos(2x) \]

Using the **chain rule** (topic of the next lecture) this result can be achieved much quicker.
Assignment: IMM1 - Tutorial 4.4